

# Generalization of order separability for free products and omnipotence of free products of groups.

Vladimir V. Yedynak

*Faculty of Mechanics and Mathematics, Moscow State University  
Moscow 119992, Leninskie gory, MSU  
edynak\_vova@mail.ru*

## Abstract

It was proved that for any finite set of elements of a free product of residually finite groups such that no two of them belong to conjugate cyclic subgroups and each of them do not belong to a subgroup which is conjugate to a free factor there exists a homomorphism of the free product onto a finite group such that the order of the image of each fixed element is an arbitrary multiple of a constant number.

*Key words:* free products, residual properties, omnipotence.

*MSC:* 20E26, 20E06.

## 1 Introduction

Order separabilities are connected with the investigation of the correlation between the orders of elements' images after a homomorphism of a group onto a finite group. For example in [1] it was proved that for each elements  $u$  and  $v$  of a free group  $F$  such that  $u$  is conjugate to neither  $v$  nor  $v^{-1}$  there exists a homomorphism of  $F$  onto a finite group such that the images of  $u$  and  $v$  have different orders. In [6] it was proved that this property is inherited by free products. This paper is devoted to the proof of the theorem that strengthens the property of order separability for the class of free products of groups.

**Theorem.** Consider the group  $G = A * B$  where the subgroups  $A$  and  $B$  are residually finite. Consider the elements  $u_1, \dots, u_n$  such that  $u_i \in G \setminus \{\cup_{g \in G} (g^{-1}Ag \cup g^{-1}Bg)\}$ ,  $u_i, u_j$  belongs to conjugate cyclic subgroups whenever  $i = j$ . Then there exists the natural number  $K$  such that for each ordered sequence  $l_1, \dots, l_n$  of natural numbers there exists a homomorphism  $\varphi$  of  $G$  onto a finite group such that the order of  $\varphi(u_i)$  is equal to  $Kl_i$ .

The property under study in this work is closely connected with omnipotence which was investigated in [2], [3] where it was shown that free groups and fundamental groups of compact hyperbolic surfaces are omnipotent. Besides all finite sets of independent elements whose orders are infinite in a Fuchsian group of the first type also satisfy the property of omnipotence [4].

**Definition.** The group  $G$  is called omnipotent if for each elements  $u_1, \dots, u_n$  such that no two of them have conjugate nontrivial powers there exists a number  $K$  such that for each ordered sequence of natural numbers  $l_1, \dots, l_n$  there exists a homomorphism  $\varphi$  of  $G$  onto a finite group such that the order of  $\varphi(u_i)$  equals  $Kl_i$ .

The familiar property was also investigated in [7] where some sufficient conditions were found for  $n$ -order separability of free products. The group  $G$  is said to be  $n$ -order separable if for a set  $S = \{s_1, \dots, s_n \mid s_i \neq h^{-1}s_j^{\pm 1}h, i \neq j\}$  of  $n$  elements of  $G$  there exists a homomorphism of  $G$  onto a finite group mapping  $S$  onto a set whose elements have pairwise different orders.

Notice that the theorem of this paper will enable to investigate the residual properties of the fundamental group of graphs of groups whose vertex groups are residually finite free products and edge groups are cyclic not belonging to subgroups conjugate to free factors of vertex groups.

## 2 Notations and Definitions

We consider that for every graph there exists a mapping  $\eta$  from the set of edges of this graph onto itself. For every edge  $e$  this mapping corresponds an edge which is inverse to  $e$ . Besides the following conditions are true:  $\eta(\eta(e)) = e$  for each  $e$ ,  $\eta$  is a bijection, for every edge  $e$  the beginning of  $e$  coincides with the end of the edge  $\eta(e)$ .

The graph is called oriented if from every pair of mutually inverse edges one of them is fixed. The fixed edge is called positively oriented and the inverse edge is called negatively oriented.

Let  $G$  be a free product of groups  $A$  and  $B$ . There exists a correspondence such that for every action of  $G$  on the set  $X$  at which both  $A$  and  $B$  act freely there exists a graph  $\Gamma$  satisfying the following properties:

- 1) for each  $c \in A \cup B$  and for each vertex  $p$  of  $\Gamma$  there exists exactly one edge labelled by  $c$  going into  $p$  and there exists exactly one edge labelled by the element  $c$  which goes away from  $p$ .
- 2) for every vertex  $p$  of  $\Gamma$  the maximal connected subgraph  $A(p)$  of  $\Gamma$  containing  $p$  whose positively oriented edges are labelled by the elements of  $A$  is the Cayley graph of the group  $A$  with generators  $\{A\}$ ; we define analogically the subgraph  $B(p)$ .
- 3) we consider that for every edge  $e$  from the first item there exists the edge inverse to  $e$  which does not bear a label; two edges with labels are not mutually inverse; edges with labels are positively oriented.

Definition 1. We say that a graph is the free action graph of the group  $G = A * B$  if it satisfies the properties 1), 2), 3).

Note that if  $\varphi$  is the homomorphism of the group  $G$  such that  $\varphi_{A \cup B}$  is the bijection then the Cayley graph  $\text{Cay}(\varphi(G); \{\varphi(A) \cup \varphi(B)\})$  of the group  $\varphi(G)$  with respect to the set of generators  $\{\varphi(A) \cup \varphi(B)\}$  is the free action graph of the group  $G$ .

Remark. In what follows appending a new edge with label to a free action graph we shall consider that it is positively oriented and the inverse edge would have been appended. And if we delete an edge with label the inverse edge would have been deleted.

If  $e$  is the edge then  $\alpha(e), \omega(e)$  are vertices which coincide with the beginning and the end of  $e$  correspondingly.

If we have the free action graph  $\Gamma$  of the group  $G$  then there exists the action of  $G$  on the set of vertices of  $\Gamma$  which is defined as follows. Let  $p$  be an arbitrary vertex of  $\Gamma$ . Then according to the definition of the free action graph for each element  $c$  from  $A \cup B$  there exist edges  $e$  and  $f$  whose labels are equal to  $c$  such that  $\alpha(e) = p, \omega(f) = p$ . In this case the action of  $c$  on  $p$  is defined as follows:  $p \circ c = \omega(e), p \circ c^{-1} = \alpha(f)$ .

Remark also that if we change the property 2) in the definition of the free action graph supposing that  $A(p)$  and  $B(p)$  are the Cayley graphs of the homomorphic images of the groups  $A$  and  $B$  correspondingly we also obtain the graph such that there exists the action of the group  $G$  on the set of its vertices. Such a graph will be referred to as an action graph of the group  $G$ .

Since there exists the action of  $G$  on the set of vertices of an action graph  $\Gamma$  there exists a homomorphism of  $G$  onto the group  $S_n$ , where  $n$  is the cardinal number of the set of vertices of the graph  $\Gamma$ . Having a group  $G$  and its action graph  $\Gamma$  we shall denote this homomorphism as  $\varphi_\Gamma$ .

If  $e$  is the positively oriented edge of the action graph, then  $\text{Lab}(e)$  is the label of  $e$ .

Definition 2. Let  $u$  be a cyclically reduced element of the group  $G$  which belongs to neither  $A$  nor  $B$  and  $\Gamma$  is the action graph of  $G$ . Fix a vertex  $p$  of  $\Gamma$ . Then  $u$ -cycle in this action graph going from  $p$  is the cycle  $R = e_1 \dots e_n$  which satisfies the following properties:

- 1) the path  $P$  is a closed path such that its beginning  $\alpha(P) = p$
- 2) consider  $u = u_1 \dots u_k$  where  $u_i \in A \cup B, u_i, u_{i+1}$  as well as  $u_1, u_k$  do not belong to one free factor simultaneously; then  $k$  divides  $n$  and the edge  $e_{ik+j}$  is positively oriented and has a label  $u_j, 1 \leq j \leq k$  (indices are modulo  $n$ )
- 3) the cycle  $P$  is the minimal cycle which satisfies properties 1), 2).

Definition 3. Suppose we have a path  $S = e_1 \dots e_n$  in the action graph. Then the label of this path is the element of the group which is equal to  $\prod_{i=1}^n \text{Lab}(e_i)'$ , where  $\text{Lab}(e_i)'$  equals either the label of  $e_i$ , if this edge is positively oriented, or  $\text{Lab}(e_i)' = \text{Lab}(\eta(e_i))^{-1}$  otherwise. We shall denote the label of the path  $S$  as  $\text{Lab}(S)$ .

Definition 4. Fix the graph  $\Gamma, p$  and  $q$  are vertices from  $\Gamma$ . Then we define the distance between  $p$  and  $q$  as  $\rho(p, q) = \min_S l(S)$ , where  $S$  is an arbitrary path connecting  $p$  and  $q, l(S)$  is the number of edges in  $S$ .

Notice that if a cycle  $S$  does not have  $l$ -near vertices then each subpath of  $S$  of length which less or equal than  $l$  is geodesic.

Definition 5. Fix an arbitrary graph and a cycle  $S = e_1 \dots e_n$  in it. For every nonnegative integer number  $l$  we shall say that  $S$  does not have  $l$ -near vertices, if for every  $i, j, i \neq j, 1 \leq i, j \leq n$  the distance between the vertices  $\alpha(e_i), \alpha(e_j)$  is greater or equal than  $\min(l+1, |i-j|, n-|i-j|)$ .

Definition 6. Suppose we have the  $u$ -cycle  $S$ . It is obvious that its label equals the  $k$ -th power of  $u$  for some  $k$ . Then we say that the length of the  $u$ -cycle  $S$  is equal to  $k$ .

Note that for the action graph  $\Gamma$  and cyclically reduced element  $u \in G \setminus \{A \cup B\}$  the order of  $|\varphi_\Gamma(u)|$  coincides with the less common multiple of lengths of

all  $u$ -cycles in the graph  $\Gamma$ . Hence if there exists a  $u$ -cycle in the action graph whose length equals  $t$  then  $|\varphi_\Gamma(u)|$  is a multiple of  $t$ .

Suppose  $u$  is an element of  $A * B$  and  $u = u_1 \cdots u_n$  is the irreducible form of  $u$ . Then the length of  $u$  is the number  $l(u) = n$ . The cyclic length  $l'(u)$  of an element  $u$  is the length of the cyclically reduced element which is conjugate to  $u$ .

### 3 Auxiliary lemmas

**Lemma 1.** Consider the group  $G = A * B$  where  $A$  and  $B$  are finite,  $l$  and  $n$  are natural numbers,  $Q$  is a finite set of elements from  $G$  which are cyclically reduced and whose lengths are greater than 1. Then for each  $v \in Q$  there exists the homomorphism  $\varphi$  of  $F$  onto a finite group such that for each  $u$  from  $Q$  the  $u$ -cycles in the Cayley graph  $\text{Cay}(\varphi(G); \{\varphi(A) \cup \varphi(B)\})$  of the group  $\varphi(G)$  do not have  $l$ -near vertices,  $|\varphi(v)| > n$ , and  $\varphi_{A \cup B \cup Q}$  — injection.

*Proof.*

For each  $q \in Q$  define the set  $L_q$  which consists of the elements from  $G$  whose length is less or equal than  $l + 2l(q) + 10$  and which do not belong to the subgroup generated by the element  $q$ . It is well known that free group are subgroup separable [5]. Besides if a group is virtually subgroup separable than it is subgroup separable (see [2] for example). Considering the above there exists the homomorphism  $\varphi_q$  of the group  $G$  onto a finite group such that  $\varphi_q(L_q) \cap \langle \varphi_q(q) \rangle$  is an empty set. There also exists the homomorphism  $\varphi'_v$  of  $G$  onto a finite group such that  $\varphi'_v(v^i) \neq 1$  where  $i = 1, \dots, n$  and  $\varphi'_v|_{A \cup B \cup Q}$  is the injection since virtually free groups are residually finite. The homomorphism  $\varphi : G \rightarrow (\times_{h \in Q} \varphi_h(G)) \times \varphi'_v(G)$ ,  $\varphi : f \mapsto \prod_{h \in Q} (\varphi_h(f)) \varphi'_v(f)$  is as required. Lemma 1 is proved.

The following statement was proved in [2]

**Lemma 2.** Consider a group  $G$  and its elements  $g_1, \dots, g_n$  possessing the property that for each  $j, 1 \leq j \leq n$ , there exist constants  $K_{j,1}, \dots, K_{j,n}$  such that for each natural  $m$  there exists a homomorphism  $\varphi_{j,m}$  of  $G$  to a finite group with the condition that  $|\varphi_{j,m}(g_k)| = K_{j,k}$  for all  $k \neq j$  and  $|\varphi_{j,m}(g_j)| = mK_{j,j}$ . Then there exists the number  $K$  such that for each ordered sequence of natural numbers  $l_1, \dots, l_n$  there exists the homomorphism  $\psi$  of  $G$  onto a finite group satisfying the property that  $|\psi(g_i)| = Kl_i$ .

### 4 Proof of the theorem

It follows from lemma 2 that that the theorem can be derived from the following proposition.

**Proposition.** Let  $G = A * B$  be a free product of residually finite groups  $A$  and  $B$ ,  $u, v_1, \dots, v_n \in G$ . Elements  $u$  and  $v_i$  do not belong to conjugate cyclic subgroups. Besides  $u$  does not belong to a subgroup which is conjugate to either  $A$  or  $B$ . Then there exist natural numbers  $L, K_1, \dots, K_n$  such that

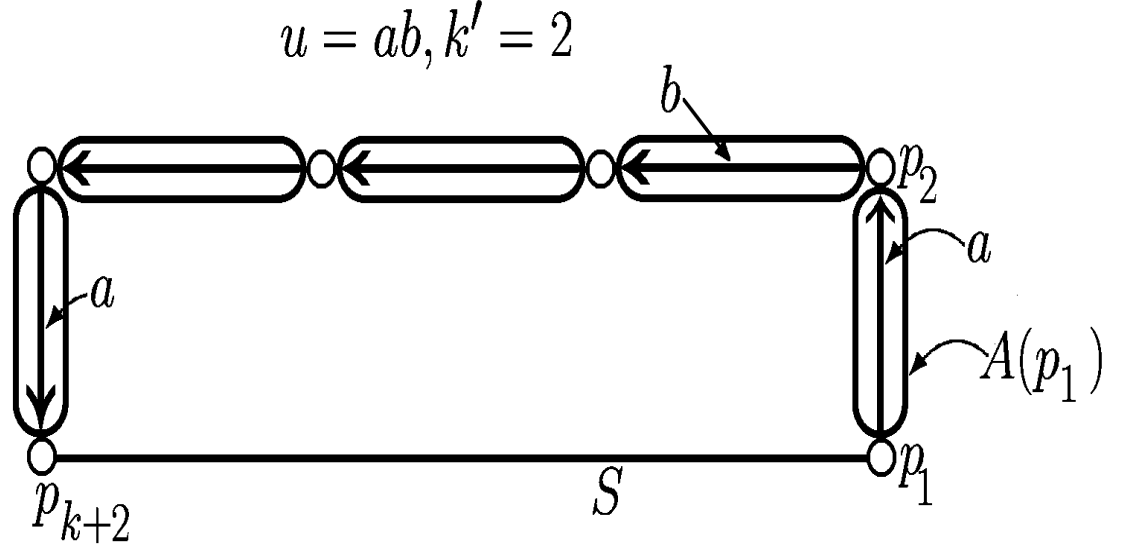


Figure 1: The graph  $\Gamma$

for each natural  $i$  there exists a homomorphism  $\varphi$  of  $G$  onto a finite group such that  $|\varphi(u)| = Li, |\varphi(v_i)| = Ki, 1 \leq i \leq n$ .

*Proof.*

Since  $A, B$  are residually finite we may consider that  $A$  and  $B$  are finite. Consider also that the elements  $u, v_1, \dots, v_n$  of  $A * B$  are cyclically reduced.

Let us to define the following notation. Consider the action graph  $\Gamma$  of the group  $K * L$ . Let  $S$  be the subset of  $\Gamma$  (e. g. vertex, edge, path, subgraph etc). Then having a set  $\Gamma_1, \dots, \Gamma_n$  of copies of  $\Gamma$  we consider that  $S^i$  denotes the subset of  $\Gamma_i$  corresponding to  $S$  in  $\Gamma$ .

Put  $s = \max_{b \in \{u, v_1, \dots, v_n\}} l(b)$ , and let  $k'$  be an arbitrary natural number such that  $k'l(u) \geq 10s$ . Put  $k = k'l(u)$ . Denote by  $P$  the set of all nonunit elements whose length is less or equal than  $10k$ . For  $Q = \{u, v_1, \dots, v_n\} \cup P$  according to lemma 1 there exists the homomorphism  $\varphi$  of  $G$  onto a finite group such that  $\varphi_{A \cup B \cup Q}$  is the injection and for each  $s \in S$  which is cyclically reduced and whose length is greater than 1 each  $s$ -cycle in the graph  $\Gamma = \text{Cay}(\varphi(G); \{\varphi(A) \cup \varphi(B)\})$  has no  $(k+4)$ -near vertices and  $|\varphi(u)| > 10k$ .

Fix a natural number  $m > 2$  whose value we shall choose later. Consider  $m$  copies of the graph  $\Gamma$ :  $\Gamma_i, 1 \leq i \leq m$ . In the graph  $\Gamma$  we fix a  $u$ -cycle  $S = e_1 \cdots e_r$ . Without loss of generality we consider that  $\text{Lab}(e_1) \in A$ . Put  $p_i = \alpha(e_i)$ ,  $\text{Lab}(e_i) = u_i$  (see Figure 1). For each  $i, 1 \leq i \leq m$ , we delete edges incident to  $p_2^i$  whose labels belong to  $A$  and delete also edges labelled by the elements of  $A$  whose begin or end points are  $p_{k+2}^i$ . For each  $i$  we shall denote the obtained graph as  $\Gamma'_i$ .

Let  $\psi$  be the bijection between the subgraphs  $A(p_1)$  and  $A(p_{k+1})$  which saves labels of edges and  $\psi(p_1) = p_{k+1}$ .

Fix an arbitrary edge  $e$  of  $\Gamma$  from the subgraph  $A(p_1)$  such that the corresponding edge  $e^i$  of  $\Gamma_i$  was deleted. Let  $q = \alpha(e), r = \omega(e)$ . For each  $i, 1 \leq i \leq m$ , if  $q \neq p_2$  we connect the vertices  $q^i$  and  $\psi(r)^{i+1}$  by the new edge  $f_i$ . If  $r \neq p_2$  we connect  $r^i$  and  $\psi(q)^{i+1}$  by the edge  $f_i$ . In both cases the label of  $f_i$  coincides with  $\text{Lab}(e)$ , besides if  $q \neq p_2$  then  $f_i$  goes away from  $q^i$  and if  $r \neq p_2$  then  $f_i$  goes into  $r^i$ .

Now we need to complement the structure of obtained graph for to get the action graph of the group  $A * B$ . But it will not be the free action graph.

For each  $i, 1 \leq i \leq m$ , let us to add one new vertex  $n_i$  to the subgraph  $\Gamma'_i$ .

Consider an arbitrary edge  $e$  from  $A(p_{k+2})$  such that the corresponding edge  $e^i$  was deleted from  $\Gamma_i$ . Put  $q = \alpha(e), r = \omega(e)$ . If  $q = p_{k+2}$  then connect the vertices  $r^i$  and  $n_i$  by the edge  $g_i$ . If  $r = p_{k+2}$  then the new edge  $g_i$  connects the vertices  $q^i$  and  $n_i$ . Put  $\text{Lab}(g_i) = \text{Lab}(e)$ . The begin point of  $g_i$  coincides with either  $n_i$  or  $q^i$ .

For each  $c \in A \cup B$  and for each vertex  $p$  of the obtained graph which is not incident to an edge with label  $c$  add a loop with label  $c$  going from  $p$ .

If we fix  $j$  then the union of the graph  $\Gamma'_j$  and  $A(p_{k+1}^j), A(p_{k+2}^j), A(p_1^j)$  is denoted by  $\Delta_j$ .

We constructed the new graph  $\Delta$  which contains subgraphs  $\Gamma'_j$  and  $\Delta_j$  and is the action graph of the group  $G$ .

In the graph  $\Delta$  the  $u$ -cycle  $S'$  going from the vertex  $p_1^1$  has the length  $(|\varphi(u)| - k')m$ . From the properties of the homomorphism  $\varphi$  it follows that  $|\varphi(u)| > 10k = 10k'l(u) > k'$ . Hence  $|\varphi_\Delta(u)| \geq (|\varphi(u)| - k')m > m$ .

Let us to prove that for each  $i$  and for each  $v_i$ -cycle  $T$  in the graph  $\Delta$  all vertices of  $T$  belong to two subgraphs  $\Delta_{j_1}, \Delta_{j_1+1}$  for some  $j_1$ . Suppose the contrary. That is we suppose that there exist pairwise different numbers  $j_1, j_2, j_3$  such that the vertices of  $T$  belong to all three subgraphs  $\Delta_{j_1}, \Delta_{j_2}, \Delta_{j_3}$ . Note that different subgraphs  $\Delta_{k_1}, \Delta_{k_2}$  has the nonempty intersection if and only if  $|k_1 - k_2| = 1$  and their intersection equals the subgraph  $A(p_1^l)$  since  $m > 2$  where  $l$  is equal to either  $k_1$  or  $k_2$ . So if  $\Delta_{j_1}, \Delta_{j_2}, \Delta_{j_3}$  contain vertices of  $T$  there exists the number  $j$  such that the subgraphs  $\Delta_j, \Delta_{j+1}, \Delta_{j+2}$  contain the vertices of  $T$  and there exists the path  $R$  which is the part of  $T$  and which belongs to  $\Delta_j \cup \Delta_{j+1} \cup \Delta_{j+2}$ ,  $R$  goes away from the vertex of  $\Delta_j$  and goes into the vertex of  $\Delta_{j+2}$  (indices are modulo  $m$ ).

From the properties of  $R$  it follows that  $R$  contains its first and the last edges  $t_j, r_j$  correspondingly such that  $t_j \in A(p_1^j), \omega(t_j) = p_{k+2}^{j+1}, r_j \in A(p_1^{j+1}), \omega(r_j) = p_{k+2}^{j+2}$ , and the rest edges of  $R$  are in  $\Delta_{j+1}$ .

Because of our supposition that  $T$  goes from  $\Delta_{j+1}$  into  $\Delta_{j+2}$  it is possible to deduce that  $R$  contains the subpath  $s_1 \cdots s_l$  such that  $s_2 \cdots s_{l-1}$  belongs to  $\Gamma'_{j+1}$  and edges  $s_1, s_l$  satisfy the following properties:  $\alpha(s_1) \in A(n_{j+1}) \cup B(p_{k+2}^{j+1}), \omega(s_l) \in A(p_1^{j+1}) \cup B(p_2^{j+1})$ .

Denote the path  $e_1^{j+1} e_2^{j+1} \cdots e_{k+1}^{j+1}$  as  $S_u$  and  $s_2 \cdots s_{l-1}$  as  $S_{v_i}$  (see Figure 2). Note that  $\rho(\alpha(S_u), \alpha(S_{v_i})) \leq 1, \rho(\omega(S_u), \omega(S_{v_i})) \leq 2$  (the function  $\rho$  is

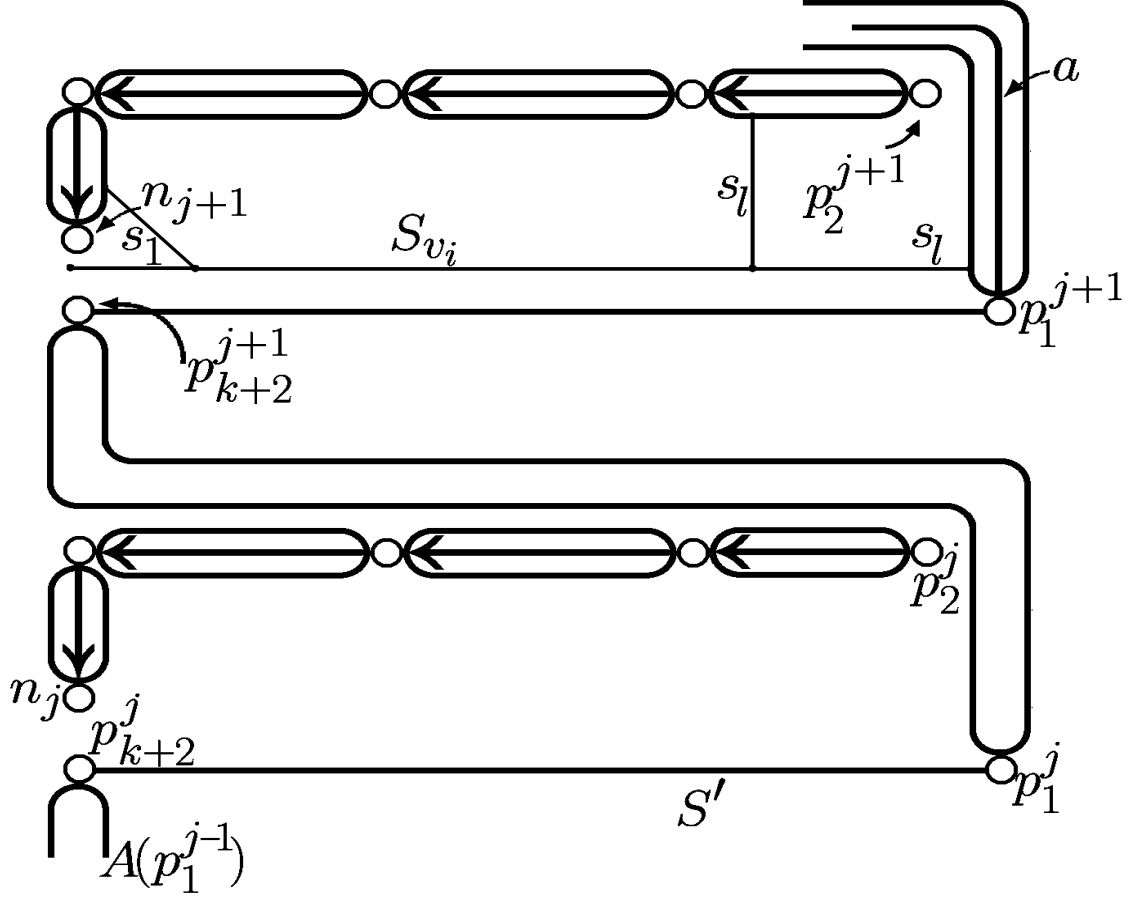


Figure 2: The graph  $\Delta$

taken with respect to  $\Gamma_i$ ). Besides  $S_{v_i}$  is a part of some  $v_i$ -cycle,  $S_u$  is a part of the  $u$ -cycle  $S'$ . Since the elements  $u, v_i$  of the group  $A * B$  do not belong to conjugate cyclic subgroups and the length of the path  $S_u$  is greater than  $10s = 10 \max_{z \in \{u, v_1, \dots, v_n\}} (l(z))$  the paths  $S_{v_i}$  and  $S_u$  are different.

Suppose that the length of the path  $S_{v_i}$  is less or equal than  $k+4 = l(S_u)+3$ . The paths  $S_{v_i}$  and  $S_u$  and perhaps several edges whose number is less than 4 compose the loop. Let  $g$  be the label of this loop. Then  $g$  is an element of the group  $G$  whose length is less or equal than  $2l(S_u) + 6 = 2k + 8 < 10k$  and  $\varphi(g) = 1$ . But this contradicts the condition on  $\varphi$  and the set  $Q$ . Thus the length of the path  $S_{v_i}$  is greater than  $k+4 = l(S_u)+3$ . By the symmetry we may also assume that the length of the path  $T \setminus S_{v_i}$  is greater than  $k+4$ : the structure of the part of  $T$  in  $\Delta_{j+2}$  is the same as in  $\Delta_{j+1}$ . But in this case  $\rho(\alpha(S_{v_i}), \omega(S_{v_i})) \leq \min(l(S_u) + 3, l(S_{v_i}), l(T \setminus S_{v_i})) = \min(k+4, l(S_{v_i}), l(T \setminus S_{v_i})) = k+4$ , since  $l(S_{v_i}), l(T' \setminus R') > k+4$ . So the  $v_i$ -cycle  $T$  containing  $S_{v_i}$  has  $(k+4)$ -near vertices. This also contradicts the conditions on  $\varphi$ .

Thus it is proved that for each  $i$  and for each  $v_i$ -cycle  $T$  in the graph  $\Delta$  there exists  $j, 1 \leq j \leq m$ , such that all vertices of  $T$  are contained in  $\Delta_j \cup \Delta_{j+1}$  (indices are modulo  $m$ ). We deduce also that each  $u$ -cycle of  $\Delta$  which does not start at  $p_1^1$  belongs to two subgraphs  $\Delta_{k_1}, \Delta_{k_1+1}$ . This can be established by the same way as it was shown that the analogical statement is true for  $v_i$ -cycles.

Now we shall denote the obtained graph  $\Delta$  for number  $m$  as  $\Delta'_m$ . Consider the set of graphs  $\Lambda_m = \Delta'_{3m}, m = 1, 2, \dots$

We shall show now that  $|\varphi_{\Lambda_m}(v_i)|$  equals some constant number  $K_i$  which does not depend on  $m$ . Let  $R_{i,m}$  be the set of lengths of all  $v_i$ -cycles of  $\Lambda_m$ . The local structure of  $\Lambda_m$  is the same: using the above notations and regarding that  $\Lambda_m$  is the union of  $\Delta_1, \dots, \Delta_{3m}$  it is obvious that the subgraphs  $\Delta_k \cup \Delta_{k+1}$  and  $\Delta_l \cup \Delta_{l+1}$  are isomorphic and do not depend on  $m$ . Hence  $R_{i,m}$  coincides with the set of lengths of  $v_i$ -cycles concentrated in  $\Delta_1 \cup \Delta_2$  and thereby  $R_{i,m} = R_{i,t}$  for all  $m, t$ . The same reasonings are true for all  $u$ -cycles of  $\Lambda_m$  except for the  $u$ -cycle whose length is the multiple of  $3m$  so  $|\varphi_{\Lambda_m}(u)| = mK$  for some constant  $K$  which does not depend on  $m$ .

Proposition is proved and therefore the theorem is also proved.

## Acknowledgements

I am grateful to Anton A. Klyachko, Ashot Minasyan, Denis Osin and Henry Wilton for valuable conversations and for information about omnipotence.

## References

1. Klyachko, A. A. Equations over groups, quasivarieties, and a residual property of a free group. *Journal of group theory* **2**: 319-327, 1999.
2. Wise, Daniel T. Subgroup separability of graphs of free groups with cyclic edge groups. *Q. J. Math.* 51, No.1, 107-129 (2000). [ISSN 0033-5606; ISSN 1464-3847]
3. Jitendra Bajpai. Omnipotence of surface groups. Masters Thesis, McGill University, 2007.
4. Henry Wilton. Virtual retractions, conjugacy separability and omnipotence. *J. Algebra* 323 (2010), pp. 323-335.
5. Marshall Hall, Jr. Coset representations in free groups. *Trans. Amer. Math. Soc.*, 67:421-432, 1948.



6. Yedynak, V. V. Separability with respect to order. *Vestnik Mosk. Univ. Ser. I Mat. Mekh.* **3**: 56-58, 2006.
7. Yedynak, V. V. Multielement order separability in free products of groups. *Communications in Algebra* 38(**3**): 3448 - 3455, 2010.